Homework 1

1. **Trapezoid Rule.** In the lecture, we saw that if f is a concave upwards function then the following is true.

$$\frac{f(x-1)+f(x)}{2} \geqslant \int_{x-1}^{x} f(t)\,\mathrm{d}t$$

   (a) **(5 points)** Prove that if f is a concave downwards function, we have

$$\frac{f(x-1)+f(x)}{2} \leqslant \int_{x-1}^{x} f(t)\,\mathrm{d}t$$

Solution.

(b) **(10 points)** Prove that, for a concave downwards function f, we have

$$f(1) + f(2) + \cdots + f(n) \leqslant \frac{f(1) + f(n)}{2} + \int_1^n f(t)\,\mathrm{d}t$$

Solution.

2. **Tight Estimations.** Provide meaningful upper-bounds and lower-bounds for the following expressions.

   (a) **(10 points)** $S_n = \sum_{i=1}^{n} \ln i$,
   Solution.

(b) **(10 points)** $A_n = n!$
Solution.

(c) **(10 points)** $B_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
Solution.

3. **Understanding Joint Distribution.** Recall that in the lectures we considered the joint distribution $(\mathbb{T}, \mathbb{B})$ over the sample space $\{1, 2, \ldots, 10\} \times \{\mathsf{T}, \mathsf{F}\}$, where $\mathbb{T}$ represents the time I wake up in the morning, and $\mathbb{B}$ represents whether I have breakfast or not. The following table summarizes the joint probability distribution.

t	b	$\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$
4	T	0.05
4	F	0.04
5	T	0
5	F	0.01
6	T	0.1
6	F	0.25
7	T	0.20
7	F	0.10
8	T	0.10
8	F	0.03
9	T	0.05
9	F	0.05
10	T	0
10	F	0.02

Calculate the following probabilities.

(a) **(5 points)** Calculate the probability that I wake up at 6 a.m. or earlier, but do not have breakfast. That is, calculate $\mathbb{P}[\mathbb{T} \leqslant 6, \mathbb{B} = \mathsf{F}]$,
Solution.

(b) **(5 points)** Calculate the probability that I wake up at 6 a.m. or earlier. That is, calculate $\mathbb{P}[\mathbb{T} \leqslant 6]$,
Solution.

(c) **(5 points)** Calculate the probability that I skip breakfast conditioned on the fact that I woke up at 6 a.m. or earlier. That is, compute $\mathbb{P}\left[\mathbb{B} = \mathsf{F} \mid \mathbb{T} \leqslant 6\right]$.
Solution.

4. **Random Walk.** There is a frog sitting at the origin $(0,0)$ in the first quadrant of a two-dimensional Cartesian plane. The frog first jumps uniformly at random along the X-axis to some point $(\mathbb{X},0)$, where $\mathbb{X} \in \{1,2,3,4,5,6\}$. Then, it jumps uniformly at random along the Y-axis to some point $(\mathbb{X},\mathbb{Y})$, where $\mathbb{Y} \in \{1,2,3,4,5,6\}$. So $(\mathbb{X},\mathbb{Y})$ represents the final position of the frog after these two jumps. Note that $\mathbb{X}$ and $\mathbb{Y}$ are two independent random variables that are uniformly distributed over their respective sample spaces.

   (a) **(5 points)** What is the probability that the frog jumps more than 4 units along the Y-axis. That is, compute $\mathbb{P}[\mathbb{Y} > 4]$.
   Solution.

   (b) **(10 points)** What is the probability that the final position of the frog is above the line $X + Y = 6$. That is compute $\mathbb{P}[\mathbb{X} + \mathbb{Y} > 6]$?
   Solution.

(c) **(10 points)** What is the probability that the frog has jumped 2 units along X-axis conditioned on the fact that its final position is above the line $X + Y = 6$? That is, compute $\mathbb{P}\left[\mathbb{X} = 2 \mid \mathbb{X} + \mathbb{Y} > 6\right]$?

5. **Coin Tossing Word Problem.** We have three (independent) coins represented by random variables $\mathbb{C}_1, \mathbb{C}_2$, and $\mathbb{C}_3$.

   (i) The first coin has $\mathbb{P}[\mathbb{C}_1 = H] = \mathbb{P}[\mathbb{C}_1 = T] = \frac{1}{2}$,

   (ii) The second coin has $\mathbb{P}[\mathbb{C}_2 = H] = \frac{3}{4}$ and $\mathbb{P}[\mathbb{C}_2 = T] = \frac{1}{4}$, and

   (iii) The third coin has $\mathbb{P}[\mathbb{C}_3 = H] = \frac{1}{4}$ and $\mathbb{P}[\mathbb{C}_3 = T] = \frac{3}{4}$.

   Consider the following experiment.

   (A) Toss the first coin. Let the outcome of the first coin-toss be ω_1.

   (B) If $\omega_1 = H$, then we toss the second coin twice. Otherwise, (i.e., if $\omega_1 = T$) toss the third coin twice. Let the two outcomes of this step be represented by ω_2 and ω_3.

   (C) Output $(\omega_1, \omega_2, \omega_3)$.

   Based on this experiment, compute the probabilities below.

   (a) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes $(\omega_1, \omega_2, \omega_3)$ are H (head)?
   Solution.

(b) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are H, conditioned on the fact that the first outcome was T?
Solution.

(c) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are different from the first outcome?
Solution.